Exercise sheet 2: Tensors, further special relativity, and coordinates

Please prepare your solutions to the following problems, ready to present in the class on **04.05.2022** at **16:00**.

1. (a) Show that the property of a tensor being symmetric or antisymmetric (in any number of its indices) is independent of the choice of basis.
   (b) Show that any rank $(0, 2)$ tensor may be represented by a combination of its completely symmetrised and completely antisymmetrised parts. Furthermore, show that this cannot be done for tensors of higher rank.
   (c) If $A_{\mu\nu}$ is symmetric, and $B^{\mu\nu}$ is antisymmetric, show that $A_{\mu\nu}B^{\mu\nu} = 0$.

2. A rocket leaves Earth in 2011 in a straight line to the red dwarf star Gliese 1062, 53 light-years away. For the comfort of the passengers, the rocket assumes a constant acceleration equal to the acceleration due to gravity g. After four rocket years, the direction of acceleration is reversed, and for another four rocket years it is decelerated with $-g$ until it reaches a standstill. Immediately afterwards it returns to earth in the same fashion: accelerated with $-g$ for four rocket years, and then decelerated with g for the same amount of time in order to land back on Earth.

   Using the values $c = 1$, $g = 3.3 \times 10^{-8}\mathrm{s}^{-1}$, 1 year $= 3 \times 10^7\mathrm{s}$, consider the following:

   (a) What is the year when the rocket returns to Earth?
   (b) Did the rocket reach its destination?
   (c) Sketch a spacetime diagram in the Earth's reference frame, including the worldline of the rocket, as well as its four-velocity and four-acceleration.

3. Consider Minkowski space $\mathbb{R}^{1,3}$ with line-element $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. For each of the following changes of coordinates, determine (in terms of the new coordinates) the line-element, and the *d'Alembertian* $\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$.

   (a) Spherical polar coordinates (t, r, θ, ϕ), defined by
   $$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \tfrac{z}{r}, \quad \tan\phi = \tfrac{y}{x};$$
   (b) Light cone coordinates (u, v, θ, ϕ), defined by the above and
   $$u = t + r, \quad v = t - r;$$
   (c) Rotating coordinates $(t, x_\omega, y_\omega, z)$, defined by
   $$x_\omega = \rho\cos(\phi - \omega t), \quad y_\omega = \rho\sin(\phi - \omega t),$$
   where $\rho = \sqrt{x^2 + y^2}$, $\tan\phi = \frac{y}{x}$, and $\omega \geq 0$ is a constant.